Coalescence cascades and special-function solutions for the continuous and discrete Painlevé equations

1998 J. Phys. A: Math. Gen. 31 5799

(http://iopscience.iop.org/0305-4470/31/27/010)

J. Phys. A: Math. Gen. **31** (1998) 5799–5810. Printed in the UK PII: S0305-4470(98)89456-0

Coalescence cascades and special-function solutions for the continuous and discrete Painlevé equations

K M Tamizhmani†, A Ramani‡, B Grammaticos§ and K Kajiwara‖

† Department of Mathematics, Pondicherry University, Kalapet, Pondicherry, 605014 India
‡ CPT, Ecole Polytechnique, CNRS, UMR 7644, 91128 Palaiseau, France
§ GMPIB (ex LPN), Université Paris VII, Tour 24-14, 5e étage 75251 Paris, France
‖ Department of Electrical Engineering, Doshisha University, Tanabe, Kyoto 610-03, Japan

Received 24 November 1997, in final form 14 April 1998

Abstract. We present a systematization of the special-function type solutions of continuous and discrete Painlevé equations. Our method is to start from P_{VI} and construct its special solutions from the solutions of the hypergeometric equation, and by coalescence obtain the lower Painlevé equations together with their special solutions. In the discrete case we study the 'symmetric', one-component, three-point mapping forms of discrete Painlevé equations starting from d-P_V.

1. Introduction

The Painlevé transcendents were introduced (by Painlevé and his collaborators) as extensions of special functions in the nonlinear domain. The Riccati equation, which is the only integrable nonlinear first-order, first-degree equation, does not introduce new transcendents since it can be reduced to a second-order linear equation through a Cole–Hopf transformation. However, at second order the problem is highly non-trivial and it led to the discovery of the new transcendents that were named after Painlevé [1]. The discovery of the Painlevé equations ($\mathbb{P}$) was due to application of the absence of movable multivaluedness requirement [2] (a property also named after Painlevé). This guaranteed the integrability of these second-order ordinary differential equations and allowed the definition of new functions through their solutions. The actual integration of the Painlevé equations was achieved, much later, through application of inverse scattering techniques [3].

While the full solution of the Painlevé equations is quite complicated and requires the consideration of (linear) integral equations, there exist also simpler solutions. These solutions exist only for special values of the parameters of the Painlevé equations and can be expressed in terms of special functions [4]. One of the important features of the special-function type solutions of the Painlevé equations is that they can be expressed in terms of Wronskians or Casorati determinants when studied in the framework of the bilinear formalism [5]. This is a feature that holds true both for continuous and discrete $\mathbb{P}$ [6]. Another type of solution also exists, the rational ones. Although their study is still lagging behind that of the special-function solutions, several recent studies [7] have started to bridge the gap.

Another interesting feature of the Painlevé equations is that they organize themselves into coalescence cascades [8]. This means that, starting from the equations containing the largest number of parameters, P_{VI}, one can obtain equations with a smaller number of parameters by successive limits corresponding to the coalescence of the singularities of the initial equation. The coalescence patterns and the degeneration of the Painlevé equations is well known and their discrete analogues have also been studied recently [9].

In this paper we shall examine the systematics of the special-function solutions in the framework of the coalescence procedure. Namely, we shall show that, starting from the highest Painlevé equation and its special solutions, one can reach the lower ones and, at the same time, construct their solutions. This will be performed for the continuous case (which is moderately well known) and also for the discrete one (which is less well known, despite recent activity in this domain). Before proceeding to the examination of specific cases, let us present the general method for obtaining the special-function solutions for continuous and discrete (symmetric) $\mathbb{P}$. The general form of a continuous Painlevé equation is

$$w'' = f(w', w, z) \tag{1.1}$$

where f is polynomial in w', rational in w and analytic in z. In order to find a solution of (1.1) in terms of special functions we assume that w is a solution of a Riccati

$$w' = Aw^2 + Bw + C \tag{1.2}$$

where A, B, C are functions of z to be determined. Substituting (1.2) into (1.1) yields an overdetermined system which allows the determination of A, B, C and *fixes the parameters* of (1.1). Equation (1.2) is subsequently linearized through the transformation

$$w = -\frac{u'}{Au}. \tag{1.3}$$

In the case of the Painlevé equations the end result is an equation of the hypergeometric family. The discrete Painlevé equations have the form

$$x_{n+1} = \frac{f_1(x_n, n) - x_{n-1} f_2(x_n, n)}{f_4(x_n, n) - x_{n-1} f_3(x_n, n)} \tag{1.4}$$

where f_i are polynomials in x_n of degree four at maximum. The solutions of (1.4) in terms of special functions proceeds through the introduction of a discrete Riccati

$$x_{n+1} = \frac{Ax_n + B}{Cx_n + D} \tag{1.5}$$

where A, B, C, D are functions of n to be determined by substituting (1.5) into (1.4). As in the continuous case, this fixes the parameters of the d-$\mathbb{P}$. The linearization of (1.5) is again obtained through a Cole–Hopf transformation

$$x_n = \left(\frac{D_n}{C_n}\right) \frac{y_{n+1} - y_n}{y_n} \tag{1.6}$$

leading to the linear equation

$$\frac{D_{n+1}}{C_{n+1}} y_{n+2} - \left(\frac{D_{n+1}}{C_{n+1}} + \frac{A_n}{C_n}\right) y_{n+1} + \left(\frac{A_n}{C_n} - \frac{B_n}{D_n}\right) y_n = 0. \tag{1.7}$$

Equation (1.7) turns out to be, in all cases concerning discrete Painlevé equations, the discrete analogue of the hypergeometric equation or one of its degenerate forms.

2. The continuous Painlevé equations' cascade

The results presented in this section are not all new in the sense that the special-function solutions of the continuous Painlevé equations have been known for some years [4]. Our main aim here is to insert these solutions into the coalescence frame. We start with the $\mathrm{P_{VI}}$ equation

$$W'' = \frac{W'^2}{2}\left(\frac{1}{W} + \frac{1}{W-1} + \frac{1}{W-Z}\right) - W'\left(\frac{1}{Z} + \frac{1}{Z-1} + \frac{1}{W-Z}\right) + \frac{W(W-1)(W-Z)}{2Z^2(Z-1)^2}\left(A - \frac{BZ}{W^2} + C\frac{Z-1}{(W-1)^2} - \frac{(D-1)Z(Z-1)}{(W-Z)^2}\right). \tag{2.1}$$

When one requires the existence of a solution given by a Riccati (1.2) the result is

$$W' = \frac{P}{Z(Z-1)}W^2 + \frac{QZ+M}{Z(Z-1)}W + \frac{N}{Z-1} \tag{2.2}$$

where the parameters P, Q, M, N are related through

$$P + Q + M + N = 0 \tag{2.3}$$

and their relation to those of $\mathrm{P_{VI}}$ is

$$A = P^2 \qquad B = N^2 \qquad C = (Q+N)^2 \qquad D = (P+Q-1)^2. \tag{2.4}$$

The condition for the existence of (2.2) is obtained when one eliminates N, P, Q from equation (2.4)

$$\epsilon_1\sqrt{A} + \epsilon_2\sqrt{B} + \epsilon_3\sqrt{C} + \epsilon_4\sqrt{D} = 1 \tag{2.5}$$

for some choice of signs ϵ_i. The linearization of (2.2) is obtained in a straightforward way through

$$W = -\frac{Z(Z-1)U'}{PU} \tag{2.6}$$

and the transformation $\zeta = (1-Z)^{-1}$ converts (2.6) to a hypergeometric equation

$$\zeta(1-\zeta)\frac{\mathrm{d}^2U}{\mathrm{d}\zeta^2} + (Q - (1-N-P)\zeta)\frac{\mathrm{d}U}{\mathrm{d}\zeta} - NPU = 0. \tag{2.7}$$

Before proceeding to the first coalescence we introduce the following convention of notation. The variables and parameters of the 'higher' equation will be represented by upper-case letters while those of the 'lower' equation will be represented by lower-case ones. The small parameter will be denoted by δ and the coalescence corresponds to the limit $\delta \to 0$.

Going from $\mathrm{P_{VI}}$ to $\mathrm{P_V}$ we have $W = w$, $Z = 1 + \delta z$, $A = a$, $B = b$, $C = d/\delta^2 + c/\delta$, $D = d/\delta^2$. One obtains thus $\mathrm{P_V}$

$$w'' = w'^2\left(\frac{1}{2w} + \frac{1}{w-1}\right) - \frac{w'}{z} + \frac{(w-1)^2}{2z^2}\left(aw - \frac{b}{w}\right) + \frac{cw}{2z} - \frac{dw(w+1)}{2(w-1)}. \tag{2.8}$$

This coalescence limit is compatible with the linearizable case provided $N = n$, $P = p$, $Q = q/\delta$ (and $M = -q/\delta - p - n$). Using (2.3), the Riccati becomes

$$w' = \frac{pw^2}{z} + \frac{(qz-p-n)w}{z} + \frac{n}{z}. \tag{2.9}$$

The parameter constraints are transformed into

$$a = p^2 \qquad b = n^2 \qquad c = 2q(n+1-p) \qquad d = q^2 \tag{2.10}$$

and the condition is readily obtained

$$\epsilon_1\sqrt{a}+\epsilon_2\sqrt{b}+\epsilon_3\frac{c}{2\sqrt{d}}=1 \tag{2.11}$$

for some choice of signs ϵ_i. Equation (2.9) is linearized through a Cole–Hopf transformation $w=-(z/p)u'/u$ to a confluent hypergeometric equation

$$u''-\left(q-\frac{1+n+p}{z}\right)u'+\frac{npu}{z^2}=0 \tag{2.12}$$

which can be transformed either to a Kummer or a Whittaker equation.

From $P_{\rm V}$ we can obtain two coalescence limits to $P_{\rm IV}$ and $P_{\rm III}$. For the first one we take $W=\delta w$, $Z=1+2\delta z$, $A=1/4\delta^4$, $B=b$, $C=-1/2\delta^4$, $D=1/4\delta^4+a/\delta^2$ and obtain $P_{\rm IV}$ in the form

$$w''=\frac{w'^2}{2w}+\frac{3w^3}{2}+4zw^2+2w(z^2+a)-\frac{2b}{w}. \tag{2.13}$$

This limit is compatible with the Riccati (2.9) which goes over to

$$w'=w^2+2zw+2n \tag{2.14}$$

provided, $P=Q=1/2\delta^2$ and $N=n$. The coefficients of $P_{\rm IV}$ (2.13), in the linearizable case are given by

$$a=n+1 \qquad b=n^2 \tag{2.15}$$

with the obvious relation

$$a+\epsilon\sqrt{b}=1 \tag{2.16}$$

for some choice of the sign ϵ. The Riccati (2.14) linearizes through $w=-u'/u$ to the Hermite equation

$$u''-2zu'+2nu=0. \tag{2.17}$$

For the second limit, to $P_{\rm III}$, we take $W=1+\delta w$, $Z=z$, $A=b/\delta^2+a/\delta$, $B=b/\delta^2$, $C=c\delta$ and $D=d\delta^2$ and obtain $P_{\rm III}$ in the non-canonical form

$$w''=\frac{w'^2}{w}-\frac{w'}{z}+\frac{bw^3}{z^2}+\frac{aw^2}{2z^2}+\frac{c}{2z}-\frac{d}{w}. \tag{2.18}$$

For the linearization of (2.18) we obtain the Riccati

$$w'=\frac{nw^2}{z}+\frac{pw}{z}+q \tag{2.19}$$

from the limit of (2.9) through $N=n/\delta$, $P=n/\delta+p$ and $Q=q\delta$. The parameters of the Riccati are related to those of $P_{\rm III}$ through

$$a=2np \qquad b=n^2 \qquad c=2q(1-p) \qquad d=q^2 \tag{2.20}$$

corresponding to the linearizability condition

$$\epsilon_1\frac{a}{2\sqrt{b}}+\epsilon_2\frac{c}{2\sqrt{d}}=1 \tag{2.21}$$

for some choice of signs. The Riccati (2.19) is linearized through the Cole–Hopf transformation $w=-(z/n)u'/u$ leading to the equation

$$zu''+(1-p)u'+nqu=0. \tag{2.22}$$

The solution of the latter is given in terms of the Bessel function J as $u=z^{p/2}J_p(2\sqrt{nqz})$.

Both P_{IV} and P_{III} go to P_{II} by coalescence. In the first case we take: $W = 2/\delta^3 + w/\delta$, $Z = -2/\delta^3 + \delta z$, $A = 2/\delta^6 + a$, $B = 4/\delta^{12}$ and obtain

$$w'' = 2w^3 + 8wz + 4a. \tag{2.23}$$

The Riccati (2.14) goes over to

$$w' = w^2 + 4z \tag{2.24}$$

provided we take $N = 2/\delta^6$ with the linearizability condition

$$a = 1. \tag{2.25}$$

The linearization of (2.24) is straightforward ($w = -u'/u$), and leads to the Airy equation

$$u'' + 4zu = 0. \tag{2.26}$$

In the second case we start from the non-canonical form of P_{III} (2.18): it turns out that this does not make any difference at the level of P_{II} apart from some unimportant coefficients. We put $W = 1 + \delta w$, $Z = 1 + \delta^2 z$, $A = -\delta^{-6}$, $B = \delta^{-6}/4 + b\delta^{-3}$, $C = \delta^{-6}$, $D = \delta^{-6}/4$ and find, at the limit $\delta \to 0$, P_{II} in the form

$$w'' = \tfrac{1}{2}w^3 + \tfrac{1}{2}wz + b. \tag{2.27}$$

The limit of the Riccati (2.19), is obtained through $Q = \delta^{-3}/2$, $N = \delta^{-3}/2$, $P = -\delta^{-3}$

$$w' = \tfrac{1}{2}(w^2 + z). \tag{2.28}$$

The linearizability condition is simply

$$2\epsilon b = 1 \tag{2.29}$$

and the linearization of the Riccati (2.28), with $w = -2u'/u$, leads again to the Airy equation

$$u'' + \frac{z}{4}u = 0. \tag{2.30}$$

Equation P_{II} degenerates to P_I through the appropriate limit. However, this coalescence does not present any interest for our purpose since it is incompatible with the existence of a Riccati equation. Indeed P_I does not possess any linearizable solution or, as a matter of fact, any particular solution.

3. Coalescence cascade for the symmetric discrete Painlevé equations

The discrete $\mathbb{P}$ we are going to work with are the ones known as 'symmetric' forms, given by the one-component mapping (1.4). We have chosen for this study the 'standard' discrete Painlevé equations which were first identified in [6]. These equations organize themselves in a coalescence cascade but the latter is not complete since the explicit form of d-P_{VI} is still missing. Moreover, this succession of discrete equations contains equations of both q-type (q-P_V, q-P_{III}) and of difference type (d-P_{IV} d-P_{II}). This is not unusual. As a matter of fact the degeneration through coalescence of q-equations quite often leads to equations of difference type.

Below, we present our results on the cascade of $\mathbb{P}$ and their special solutions starting from q-P_V

$$(X_{n+1}X_n - 1)(X_nX_{n-1} - 1) = \frac{(X_n - U)(X_n - 1/U)(X_n - V)(X_n - 1/V)}{(X_n/P - 1)(X_n/Q - 1)} \tag{3.1}$$

where U, V are constants and P, Q are proportional to Λ^n.

The most efficient way to apply the method described in the introduction (as we have explained in previous works) is to seek a factorization of the equation into

$$X_{n+1}X_n - 1 = \frac{(X_n - U)(X_n - V)}{UV(X_n/P - 1)} \tag{3.2a}$$

$$X_n X_{n-1} - 1 = \frac{UV(X_n - 1/U)(X_n - 1/V)}{X_n/Q - 1}. \tag{3.2b}$$

Equation (3.2*a*) can be rewritten as a homographic mapping (discrete Riccati)

$$X_{n+1} = \frac{X_n - U - V + UV/P}{UV(X_n/P - 1)} \tag{3.3}$$

and by up-shifting (3.2*b*) and solving for X_{n+1} we obtain the *same* homographic mapping *provided*

$$UVQ\Lambda = P. \tag{3.4}$$

If this condition holds then q-P$_{\mathrm{V}}$ possesses solutions linearizable through the discrete Riccati (3.3). The linearization of the latter was given in [10] where we have shown that X can be expressed in terms of discrete confluent hypergeometric functions. Indeed, putting $X = R/S$ we find that $R_n = P(S_n - S_{n+1})$ and S obeys the discrete confluent hypergeometric equation

$$S_{n+2} + \left(\frac{1}{\Lambda UV} - 1\right) S_{n+1} + \frac{1}{\Lambda}\left(\frac{1}{U} - \frac{1}{P}\right)\left(\frac{1}{V} - \frac{1}{P}\right) S_n = 0 \tag{3.5}$$

where we are reminded that $P \propto \Lambda^n$. The continuous limit of (3.3) should coincide with the Riccati obtained for P$_{\mathrm{V}}$ in section 2. As a matter of fact, implementing the continuous limit through $\Lambda = 1+\epsilon$, $U = 1+\epsilon\nu$, $V = -1-\epsilon\rho$, $P = (1/\epsilon+\mu)/Z$ and $Q = (-1/\epsilon+\mu)/Z$, one does not obtain, at $\epsilon \to 0$, the same Riccati equation. This is due to the fact that one has also to transform the dependent variable. We have indeed $X = (1+W)/(1-W)$ where W is the variable that goes over to that of P$_{\mathrm{V}}$ in the continuous limit. Using this transformation one obtains P$_{\mathrm{V}}$ with $a = \rho^2$, $b = \nu^2$, $c = -8\mu$ and $d = 4$. The linearization constraint (3.4) becomes at the limit $\rho+\nu+1 = 2\mu$ which is consistent with the continuous condition (2.11). A computation of the continuous limit of the Riccati for W yields

$$W' = \frac{\rho W^2}{Z} + \frac{(2Z - \rho + \nu)W}{Z} - \frac{\nu}{Z} \tag{3.6}$$

which is exactly (2.9) with $p = \rho$, $n = -\nu$ and $q = 2$ which is in accordance with (2.10).

We proceed now to the first coalescence q-P$_{\mathrm{V}} \to$ d-P$_{\mathrm{IV}}$, using the same convention of upper/lower-case symbols as in continuous $\mathbb{P}$. Putting $X = 1+\delta x$ and $U = 1+\delta a$, $V = 1+\delta b$, $P = 1+\delta(z+c)$ and $Q = 1+\delta(z-c)$, i.e. $\Lambda = 1+\delta\alpha$ such that $z = \alpha n+\beta$, we obtain, at $\delta \to 0$ the discrete P$_{\mathrm{IV}}$

$$(x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{(x_n^2 - a^2)(x_n^2 - b^2)}{(x_n - z)^2 - c^2}. \tag{3.7}$$

The linearization is again obtained by factorization

$$x_{n+1} + x_n = \frac{(x_n - a)(x_n - b)}{x_n - z - c} \tag{3.8a}$$

$$x_n + x_{n-1} = \frac{(x_n + a)(x_n + b)}{x_n - z + c} \tag{3.8b}$$

and the two equations are compatible if the following constraint holds

$$a + b + \alpha = 2c \tag{3.9}$$

which is exactly what would result from the coalescence limit of (3.4). We must now check that:

(*a*) equation (3.8*a*) is indeed the discrete Riccati one obtains from q-$\mathrm{P_V}$ through coalescence, and

(*b*) its continuous limit is the Riccati for the linearizable solutions of $\mathrm{P_{IV}}$.

Both calculations are straightforward. It suffices to implement the coalescence limit on (3.3) in order to obtain (3.8*a*). Moreover, the continuous limit of (3.8*a*), obtained through $c = 1/\epsilon$, $b = 2/\epsilon$, $a = \nu\epsilon$ with $x = w$ and z_n going over to the continuous variable z, is

$$w' = w^2 - 2zw - 2\nu \tag{3.10}$$

i.e. the equation we obtained by linearizing $\mathrm{P_{IV}}$. Equation (3.8*a*) has been shown to be solvable in terms of the discrete analogues of Hermite functions [11].

The second coalescence one can obtain from q-$\mathrm{P_V}$ is that to q-$\mathrm{P_{III}}$. It is based on the limit $X = x/\delta$, $\Lambda = \lambda$, $U = a/\delta$, $V = \delta/b$, $P = p/\delta$ and $Q = q/\delta$ leading to

$$x_{n+1}x_{n-1} = \frac{(x_n - a)(x_n - b)}{(x_n/p - 1)(x_n/q - 1)} \tag{3.11}$$

where $p = p_0\lambda^n$ and $q = q_0\lambda^n$. The linearization is again given by a factorization [12]

$$x_{n+1} = \frac{b}{a}\frac{(x_n - a)}{(x_n/p - 1)} \tag{3.12a}$$

$$x_{n-1} = \frac{a}{b}\frac{(x_n - b)}{(x_n/q - 1)} \tag{3.12b}$$

and the compatibility of (3.12*a*) and (3.12*b*) is

$$bp = aq\lambda \tag{3.13}$$

in which case (3.12) is solved in terms of discrete Bessel functions. Again, (3.13) is exactly the limit of (3.4) under the coalescence procedure. In perfect parallel to the q-$\mathrm{P_V} \to$ d-$\mathrm{P_{IV}}$ case we can show that (3.12*a*) is the coalescence limit of (3.3). As far as the continuous limit is concerned we must take into account the non-canonical character of (2.18). Putting $x = w$, $\lambda = 1 + \epsilon$, $b = -a + \epsilon c$, $p_0 = -1$ and $q_0 = 1 + \epsilon d$ we find for the continuous limit of q-$\mathrm{P_{III}}$ the (again non-canonical)

$$w'' = \frac{w'^2}{w} - \frac{w'}{z} + w^3 + \frac{dw^2}{z} - \frac{c}{z^2} - \frac{a^2}{wz^2} \tag{3.14}$$

where $z = n\epsilon$. The continuous limit of (3.12*a*) is the Riccati

$$w' = -w^2 - \frac{c}{a}\frac{w}{z} - \frac{a}{z} \tag{3.15}$$

which, with $w = u'/u$, is linearized to $zu'' + cu'/a + au = 0$. The latter is solvable, as expected, in terms of Bessel functions $u = z^{(a-c)/2a}J_{1-c/a}(2(az)^{1/2})$.

Two coalescence limits remain to be considered, those of d-$\mathrm{P_{IV}}$ and q-$\mathrm{P_{III}}$ to d-$\mathrm{P_{II}}$. In the first case we put $X = 1+\delta x$, $A = 1+\delta$, $B = -1+\delta$, $Z = 1-\delta^2 z/4$ and $C = \delta-\delta^2 a/4$ and find that

$$x_{n+1} + x_{n-1} = \frac{z_n x_n + a}{1 - x_n^2}. \tag{3.16}$$

Starting from (3.9) we implement the coalescence limit and find that the linearizability condition of d-$\mathrm{P_{II}}$ is

$$a = \frac{\alpha}{2} \tag{3.17}$$

while the homographic mapping (3.8) reduces to

$$x_{n+1} + 1 = \frac{a + z_n}{2(1 - x_n)} \tag{3.18a}$$

$$x_{n-1} - 1 = \frac{a - z_n}{2(1 + x_n)}. \tag{3.18b}$$

This leads to the linearization of d-$\mathrm{P_{II}}$. It can be checked that the continuous limit of (3.18), (obtained by $x = \epsilon w$ and $a = 2\epsilon^3$, while the discrete variable z_n is related to the continuous variable z through $z_n = 2 + 4\epsilon^2 z$), coincides with the Riccati (2.24) from $\mathrm{P_{II}}$.

In a similar way, one can work out the coalescence q-$\mathrm{P_{III}} \to$ d-$\mathrm{P_{II}}$. We start first by transforming (3.16) by $X = Y/Z$, where $Z = \Lambda^n$, to

$$Y_{n+1}Y_{n-1} = \frac{P_0 Q_0 (Y_n - A/Z)(Y_n - B/Z)}{(Y_n - P_0)(Y_n - Q_0)}. \tag{3.19}$$

Next, we introduce $Y = 1+\delta x$, $P_0 = 1+\delta$, $Q_0 = 1-\delta$, $A = 1+\delta+\delta^2 a/2$, $B = 1-\delta-\delta^2 a/2$ and $\Lambda = 1-\delta^2\alpha/2$ leading to $Z = 1-z\delta^2/2$ and find in the limit d-$\mathrm{P_{II}}$ precisely in the form (3.16). Again the linearizability condition, resulting from the limit of (3.13) is identical to (3.17). The discrete Riccati is also identical to (3.18). As we have shown in [13] the solution of the latter (and thus of d-$\mathrm{P_{II}}$ for $a = \alpha/2$) is given in terms of discrete Airy functions.

4. The coalescence of the Wronskian–Casorati solutions

In the previous sections we have shown that the coalescence procedure can be applied to the cases where the Painlevé equations are solvable through linearization in terms of special functions. This study was limited to only the lowest of the linearizable cases, i.e. the cases where the Painlevé equation can be linearized through a Cole–Hopf transformation (or, equivalently, when it can be reduced to a Riccati). However, it is well known that higher solutions of the special-function type do exist for values of the parameters that are simply related to those of the lowest, ‘fundamental’, solutions and the solution of the Painlevé equation can be expressed as a ratio of $(N \times N)$ Wronskian or Casorati determinants (the elements of which are special functions). The fundamental solution is the one obtained in the case of $N = 1$ determinants. It is thus interesting to investigate the fate of these higher Wronskian–Casorati solutions under the coalescence limit. In the following we do not examine the full cascade from $\mathrm{P_{VI}}$ to $\mathrm{P_{II}}$. As a matter of fact, knowledge of the Casorati expressions is still fragmentary in the case of discrete Painlevé equations. We rather limit ourselves to the coalescence $\mathrm{P_{III}} \to \mathrm{P_{II}}$ in both the continuous and discrete setting.

We start from $\mathrm{P_{III}}$, which is given here in a form slightly different from that of (2.18), but which is more convenient

$$W'' = \frac{W'^2}{W} + \mathrm{e}^{2Z}\left(W^3 - \frac{1}{W}\right) + \mathrm{e}^{Z}(AW^2 + B) \tag{4.1}$$

(note the change of the independent variable).

The higher linearizability condition which allows the solution of $\mathrm{P_{III}}$ to be expressed in terms of τ-functions is

$$A + B = 2(2N + 1) \tag{4.2}$$

for an integer N which in the case $N = 0$ reduces to (2.21). Putting $A = -2\nu + 2N$ and $B = 2\nu + 2N + 2$ we find that W can be given as [14]

$$W = \mathrm{e}^{-Z}\left(N + \nu + \frac{\mathrm{d}}{\mathrm{d}Z}\ln\left(\frac{\tau_N^{\nu+1}}{\tau_{N+1}^{\nu}}\right)\right) \tag{4.3}$$

where τ_N^ν is just the $(N \times N)$ Wronskian determinant

$$\tau_N^\nu = \begin{vmatrix} J_\nu & \frac{\mathrm{d}}{\mathrm{d}Z}J_\nu & \dots & \frac{\mathrm{d}^N}{\mathrm{d}Z^N}J_\nu \\ \frac{\mathrm{d}}{\mathrm{d}Z}J_\nu & \ddots & & \vdots \\ \vdots & & & \\ \frac{\mathrm{d}^N}{\mathrm{d}Z^N}J_\nu & \dots & & \frac{\mathrm{d}^{2N}}{\mathrm{d}Z^{2N}}J_\nu \end{vmatrix}. \tag{4.4}$$

Here, J_ν are Bessel functions or, more precisely, $J_\nu(e^Z)$ is a solution of the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}Z^2}J_\nu + (\mathrm{e}^{2Z} - \nu^2)J_\nu = 0. \tag{4.5}$$

In order to proceed to the coalescence limit, we introduce the following transformation of the independent variable $\mathrm{e}^Z = 1/\delta^3 + z/\delta$ together with $\nu = 1/\delta^3$. Thus, for $\delta \to 0$ and $\nu \to \infty$, equation (4.5) becomes

$$\frac{\mathrm{d}^2 J}{\mathrm{d}z^2} + 2zJ = 0 \tag{4.6}$$

i.e. the Airy equation. Simultaneously, we have for the dependent variable: $W = 1 + \delta w$ and by taking $A = -2/\delta^3 + 2N$, $B = 2/\delta^3 + 2N + 2$, we obtain for w, equation $\mathrm{P_{II}}$ in the form

$$w'' = 2w^3 + 4zw + 4N + 2. \tag{4.7}$$

The only subtle point remaining to prove is that the limit of (4.3) leads indeed to the Wronskian solution of (4.7). As a matter of fact, (4.3) is expressed in terms of τ^ν and $\tau^{\nu+1}$ which involve J_ν and $J_{\nu+1}$, respectively. These Bessel functions (of argument e^Z) are related through $J_{\nu+1} = \mathrm{e}^{-Z}(\nu J_\nu - \mathrm{d}J_\nu/\mathrm{d}Z)$. Introducing the transformation of independent variables from Z to z, we find that $J_{\nu+1} = J_\nu + \mathcal{O}(\delta)$. Thus, at the limit $\delta \to 0$, τ^ν and $\tau^{\nu+1}$ are expressed in terms of the *same* solution of (4.6). In conclusion, the coalescence of the Wronskian solutions of $\mathrm{P_{III}}$ lead indeed to the Wronskian solutions of $\mathrm{P_{II}}$ [15]

$$w = \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\tau_N}{\tau_{N+1}} \tag{4.8}$$

for all integers N, where τ-functions are given by (4.4), where the entries are in terms of the solution J of the Airy equation (4.6).

We turn now to the case of the coalescence from q-$\mathrm{P_{III}}$ to d-$\mathrm{P_{II}}$. We start from q-$\mathrm{P_{III}}$ in the form

$$X_{n+1}X_{n-1} = \frac{PQ(X_n - AZ)(X_n - BZ)}{(X_n - P)(X_n - Q)} \tag{4.9}$$

where $Z = \Lambda^n$ and A, B, P, Q are constants.

The higher linearizability condition is

$$AQ = BP\Lambda^{1+2N} \tag{4.10}$$

where N is an integer. In the case $N = 0$ one obtains the linearizability condition which is nothing but (3.13) with $\Lambda = 1/\lambda$. For $N = 0$ we put

$$X_n = P + \frac{J_{n+1}}{J_n} \tag{4.11}$$

and find for J the equation

$$J_{n+2} + (P - Q)J_{n+1} + Q(AZ - P)J_n = 0. \tag{4.12}$$

The function J is characterized by one parameter ν which can be expressed in terms of the parameters of q-P$_{\mathrm{III}}$ as $P/Q = -\Lambda^{\nu}$. In fact, a simple expression for ν exists for any value of N

$$\frac{AP}{BQ} = \Lambda^{1+2\nu}. \tag{4.13}$$

Equation (4.12) is a discrete form of the Bessel equation. One can easily derive the contiguity relations for the discrete Bessel function $J_n^{(\nu)}$. We find

$$J_n^{(\nu+1)} = \frac{1}{\sqrt{Z}}\left(J_n^{(\nu)} + \frac{1}{P}J_{n+1}^{(\nu)}\right) \tag{4.14a}$$

$$J_n^{(\nu-1)} = \frac{1}{\sqrt{Z}}\left(J_n^{(\nu)} - \frac{1}{Q}J_{n+1}^{(\nu)}\right) \tag{4.14b}$$

where $(\nu+1)$ and $(\nu-1)$ are associated with values of the parameters $A\sqrt{\Lambda}$, $B/\sqrt{\Lambda}$, $P\sqrt{\Lambda}$, $Q/\sqrt{\Lambda}$ and $A/\sqrt{\Lambda}$, $B\sqrt{\Lambda}$, $P/\sqrt{\Lambda}$, $Q\sqrt{\Lambda}$, respectively.

For generic N, the form of X is given in terms of τ-functions as [16]

$$X_n = P + \frac{\tau_{n+1}^{(\nu,N+1)}\tau_n^{(\nu+1,N)}}{\tau_n^{(\nu,N+1)}\tau_{n+1}^{(\nu+1,N)}} \tag{4.15}$$

where $\tau_n^{(\nu,N)}$ is given by the $(N \times N)$ Casorati determinant of discrete Bessel functions

$$\tau_n^{(\nu,N)} = \begin{vmatrix} J_n^{(\nu)} & J_{n+1}^{(\nu)} & \cdots & J_{n+N-1}^{(\nu)} \\ J_{n+2}^{(\nu)} & \ddots & & \vdots \\ \vdots & & & \\ J_{n+2N-2}^{(\nu)} & \cdots & & J_{n+3N-3}^{(\nu)} \end{vmatrix}. \tag{4.16}$$

We now proceed to the coalescence limit and introduce $X = 1+\delta x$, $P = 1-\delta$, $Q = 1+\delta$, $A = 1 - \delta - \delta^2 a/2$, $B = 1 + \delta + \delta^2 a/2$ and $\Lambda = 1 + \delta^2\alpha/2$ leading to $Z = 1 + z\delta^2/2$. Moreover, since $P/Q = -\Lambda^{\nu}$, we have for ν at leading order $\nu = 2\mathrm{i}\pi/(\alpha\delta^2)$, which means that ν diverges at the limit $\delta \to 0$. We thus obtain d-P$_{\mathrm{II}}$ in the form of (3.16) where the parameter a resulting from the limit of (4.10) is

$$a = \alpha(N + 1/2). \tag{4.17}$$

The coalescence of the Bessel equation (4.12) leads first to

$$J_{n+2} - 2\delta J_{n+1} + \delta^2(z - a)J_n/2 = 0 \tag{4.18}$$

and we absorb the δ factors through a gauge transformation $J_n = \delta^n K_n$. We thus find for K

$$K_{n+2} - 2K_{n+1} + (z - a)K_n/2 = 0 \tag{4.19}$$

which is precisely the discrete form of the Airy equation. In the coalescence process ν disappears from the limit equation. Note that during the limiting process the solution K_n

could *a priori* have kept the memory of the value of ν (through the value of $J^{(\nu)}$ it came from). However, this is not the case. In fact, since $J_n = \delta^n K_n$ the term $J_{n+1}^{(\nu)}$ becomes negligible in the contiguity relations (4.14). Moreover, since Z goes to 1 at lowest order, K_n are indeed independent of the index ν. By taking the limit of (4.15) we obtain the Casorati determinant solution of d-$\mathrm{P_{II}}$. Starting from (4.16) and expressing J_n in terms of K_n we obtain τ functions of the same form (where J_n are replaced by K_n) and a global factor that is some power of δ. However, it turns out that when we compute the ratio appearing in (4.15) all δ but one drop out. We thus have

$$X_n = 1 - \delta + \delta \frac{\tau_{n+1}^{(N+1)} \tau_n^{(N)}}{\tau_n^{(N+1)} \tau_{n+1}^{(N)}} \tag{4.20}$$

and since $X = 1 + \delta x$ we see that the solution of d-$\mathrm{P_{II}}$ is given by [17]

$$x_n = -1 + \frac{\tau_{n+1}^{(N+1)} \tau_n^{(N)}}{\tau_n^{(N+1)} \tau_{n+1}^{(N)}} \tag{4.21}$$

for all integers N.

5. Conclusion

In this paper we have examined the effect of coalescence on the special solutions of the continuous and discrete Painlevé equations. We have shown that these special solutions follow exactly the same cascade as the Painlevé equations corresponding to the degeneration pattern of special functions:

hypergeometric → confluent hypergeometric → {Weber, Bessel} → Airy.

We must emphasize here that this degeneration pattern was not previously established in the discrete case and that this work is the first to stress this parallel between continuous and discrete special functions.

As stated in the introduction, the Painlevé equations possess another class of special solutions, namely the rational ones. It would be interesting to study the effect of coalescence on these solutions. However, this study cannot be performed before the whole landscape of rational Casorati expressions is explored and we must hasten to point out that, to date, the results in this direction are indeed scant. The main difference between special-function and rational solutions is that usually the fundamental rational solution is an almost trivial one. Thus, the study of the effect of coalescence on rational solutions should be performed on higher Casorati solutions.

In the case of discrete Painlevé equations we have limited ourselves here to the class that has been thoroughly studied, the symmetric (one-component) mappings. The asymmetric (two-component) class would be equally interesting to analyse. However, in this case a considerable amount of preliminary work is needed since the special-function type solutions of this family have not been completely studied yet. Another interesting feature of the discrete case is that more than one coalescence cascade exists and the degeneration pattern can be more complicated than that of the symmetric case. As suggested from the present results on the symmetric case, the degeneration of the special-function type solutions for asymmetric d-$\mathbb{P}$ should follow this pattern and would thus present considerable interest from the point of view of discrete special functions. We expect to return to this problem in some future work.

Acknowledgments

The financial help of the CEFIPRA, through contract 1201-1, is gratefully acknowledged. K Kajiwara was supported by the grant-in-aid for the Encouragement of Young Scientists, The Ministry of Education, Science, Sports and Culture of Japan, No. 09740164.

References

[1] Painlevé P 1902 *Acta. Math.* **25** 1
[2] Ramani A, Grammaticos B and Bountis A 1989 *Phys. Rep.* **180** 159
[3] Ablowitz M J and Segur H 1997 *Phys. Rev. Lett.* **38** 1103
[4] Gromak V A and Lukashevich N A 1990 *The Analytic Solutions of the Painlevé Equations* (Minsk: Universitetskoye Publishers) (in Russian)
[5] Okamoto K 1986 *Ann. Math.* **146** 337
Okamoto K 1987 *Japan. J. Math.* **13** 47
Okamoto K 1987 *Funkc. Ekv.* **30** 305
[6] Ramani A, Grammaticos B and Hietarinta J 1991 *Phys. Rev. Lett.* **67** 1829–32
[7] Kajiwara K and Ohta Y 1996 *J. Math. Phys.* **37** 4693
Kajiwara K, Yamamoto K and Ohta Y 1997 *Phys. Lett.* **232A** 189
Noumi M and Yamada Y 1998 Symmetries in the fourth Painlevé equation and Okamoto polynomials *Nagoya Math. J.* to appear
Kajiwara K and Ohta Y 1998 *J. Phys. A: Math. Gen.* **31** 2431
Hietarinta J and Kajiwara K 1996 Rational solutions to d-P_{IV} *Proc. of the 2nd SIDE workshop (Canterbury, 1996)* to appear
[8] Ince E L 1956 *Ordinary Differential Equations* (New York: Dover)
[9] Grammaticos B and Ramani A 1996 *Physica* **228A** 160
[10] Tamizhmani K M, Grammaticos B, Ramani A and Ohta Y 1996 *Lett. Math. Phys.* **38** 289
[11] Tamizhmani K M, Grammaticos B and Ramani A 1993 *Lett. Math. Phys.* **29** 49
[12] Grammaticos B, Nijhoff F W, Papageorgiou V G, Ramani A and Satsuma J 1994 *Phys. Lett.* **185A** 446
[13] Ramani A and Grammaticos B 1992 *J. Phys. A: Math. Gen.* **25** L633
[14] Kajiwara K, Ohta Y and Satsuma J 1994 *RIMS Kokyuroku* **889** 124
[15] Okamoto K 1986 *Math. Ann.* **275** 221
[16] Kajiwara K, Ohta Y and Satsuma J 1995 *J. Math. Phys.* **35** 4162
[17] Kajiwara K, Ohta Y, Satsuma J, Grammaticos B and Ramani A 1994 *J. Phys. A: Math. Gen.* **27** 915